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PACKING DIRECTED CIRCUITS FRACTIONALLY

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Let G be a digraph, and let $k \geq 1$, such that no "fractional" packing of directed circuits of G has value $> k$, when every vertex is given "capacity" 1. We prove there is a set of $O(k \log k \log\log k)$ vertices meeting all directed circuits of G.

1. Introduction

A well-known conjecture of Younger [5] asserts the following.

(1.1). (Conjecture) *For every integer $k \geq 0$ there exists $n \geq 0$ such that for every digraph G, either*
 (i) *there are k directed circuits of G, mutually vertex-disjoint, or*
 (ii) *there exists $X \subseteq V(G)$ with $|X| \leq n$ such that $X \cap V(C) \neq \emptyset$ for every directed circuit C of G.*

(Digraphs in this paper are finite; $V(G)$ denotes the set of vertices of G.)

This is trivial for $k \leq 1$, and has recently been proved for $k=2$ by McCuaig [4], when we may take $n=3$. For $k \geq 3$, however, (1.1) remains open.

Our objective in this paper is to prove a fractional version of (1.1). A *fractional circuit packing* in a digraph G is a function q assigning a non-negative rational $q(C)$ to every directed circuit C of G, such that for every vertex v of G,

$$\Sigma(q(C) : v \in V(C)) \leq 1.$$

We define the *value* of q to be $\Sigma q(C)$, summed over all directed circuits C. Clearly, there is a (0, 1)-valued fractional circuit packing of value k if and only if (1.1)(i) holds. Our main result is the following.

(1.2). *Let G be a digraph, and let $k \geq 1$ be a real number such that every fractional circuit packing in G has value $\leq k$. Then there exists $X \subseteq V(G)$ with*

$$|X| \leq 4k \log(4k) \log\log_2(4k)$$

Mathematics Subject Classification (1991): 05 C 20, 05 C 38, 05 C 70 *such that* $X \cap V(C) \neq \emptyset$ *for every directed circuit* C *of* G.

[$\log(n)$ denotes the logarithm of n to base e, and $\log_2(n)$ to base 2.]

By linear programming duality, (1.2) is equivalent to the following. ($\mathbf{R}_+$ is the set of non-negative real numbers.)

(1.3). *Let* G *be a digraph, and let* $w: V(G) \to \mathbf{R}_+$ *be a function such that*

$$\Sigma(w(v) : v \in V(C)) \geq 1$$

for every directed circuit C *of* G. *Let* $k \geq 1$ *be real, and let*

$$\Sigma(w(v) : v \in V(G)) \leq k.$$

Then there exists $X \subseteq V(G)$ *with*

$$|X| \leq 4k \log(4k) \log\log_2(4k)$$

such that $X \cap V(C) \neq \emptyset$ *for every directed circuit* C *of* G.

To prove (1.3), we may assume by a continuity argument that $w(v) > 0$ for every vertex v, and that each $w(v)$ is rational. Let $n > 0$ be an integer such that $w'(v) = nw(v)$ is an integer for every vertex v. Construct a new digraph H as follows. For each $v \in V(G)$, let $(v,1), \ldots, (v, w'(v))$ be new vertices in H, where (v,i) is adjacent to $(v, i+1)$ for $1 \leq i < w'(v)$. For each edge of G with tail u and head v, let there be an edge of H with tail $(u, w'(u))$ and head $(v,1)$. Then H has the properties that:

(i) $|V(H)| = \sum_{v \in V(G)} w'(v) = \sum_{v \in V(G)} nw(v) \leq nk$,

(ii) for every directed circuit C' of H there is a directed circuit C of G so that

$$|V(C')| = \sum_{v \in V(C)} nw(v) \geq n \geq k^{-1}|V(H)|.$$

Consequently, from (1.4) stated below and proved in the next section, there exists $X \subseteq V(H)$ with

$$|X| \leq 4k \log(4k) \log\log_2(4k),$$

meeting every directed circuit of H, and so the same holds for G; that is, (1.3) is true. This shows that in order to prove (1.3) it suffices to prove the following.

(1.4). *Let* G *be a digraph, and let* $k \geq 1$ *be real, such that every directed circuit of* G *has length* $\geq k^{-1}|V(G)|$. *Then there exists* $X \subseteq V(G)$ *with*

$$|X| \leq 4k \log(4k) \log\log_2(4k)$$

such that $X \cap V(C) \neq \emptyset$ *for every directed circuit* C *of* G.

We shall prove (1.4) in the next section.

2. The main proof

The idea of the proof of (1.4) is quite simple. We proceed by induction on $|V(G)|$. Since every directed circuit of G has length $\geq |V(G)|/k$, there are vertices u,v such that every directed $u-v$ path has length (almost) $|V(G)|/k$, and hence there are disjoint non-null sets $X_1,\ldots,X_n$ with union $V(G)$, such that no edge of G has tail in X_i and head in X_j for $j \geq i+2$, where n is roughly $|V(G)|/k$. Since $X_1,\ldots,X_n$ have cardinality $\leq k$ on average, one of them has cardinality $\leq k$, and we can choose one (X_i say) roughly in the middle of the sequence with $|X_i|$ not much bigger than k. We delete X_i and apply the inductive hypothesis to the restrictions of G to $X_1 \cup \ldots \cup X_{i-1}$ and to $X_{i+1} \cup \ldots \cup X_n$. Both the latter have at most about $\left(1-\frac{1}{2k}\right)|V(G)|$ vertices, and the result follows after some calculations. This easily yields (1.4) with the bound on $|X|$ replaced by a quadratic function of k. However, it seems of interest to prove the sharpest version of (1.4) that we can, and so we shall use a numerically more careful argument.

Let us say μ is *suitable* if μ is a real-valued function with domain the set of positive real numbers, such that

(i) μ is non-negative,
(ii) $\mu(x) \geq x$ for all $x \geq 1$,
(iii) $\mu(x+y) \geq \mu(x)+\mu(y)$ for all $x,y>0$ (and consequently μ is monotone non-decreasing),
(iv) if $y \geq x \geq \frac{1}{4}$ and $x+y \geq 1$ then

$$\mu(x+y)-\mu(x)-\mu(y) \geq 4x \log\left(1+\frac{y}{x}\right)\log\log_2 4(x+y).$$

The reason for interest in suitable μ is that we shall prove the following.

(2.1). *Let μ be suitable. Let G be a digraph, and let $k>0$ be a real number such that every directed circuit has length $\geq k^{-1}|V(G)|$. Then there exists $X \subseteq V(G)$ with $|X| \leq \mu(k)$ such that $X \cap V(C) \neq \emptyset$ for every directed circuit C of G.*

(1.4) follows from (2.1) because of the following.

(2.2). *Let $\mu(x)=0$ for $0<x<1$, and $\mu(x)=4x\log(4x)\log\log_2(4x)$ for $x \geq 1$. Then μ is suitable.*

Proof. The first three conditions are easily verified. For the last, let $y \geq x \geq \frac{1}{4}$, with $x+y \geq 1$. Since

$$\mu(x+y)-\mu(x)-\mu(y) = x\left[\frac{\mu(x+y)}{x+y}-\frac{\mu(x)}{x}\right]+y\left[\frac{\mu(x+y)}{x+y}-\frac{\mu(y)}{y}\right]$$

and $\frac{\mu(x+y)}{x+y}-\frac{\mu(y)}{y} \geq 0$, it suffices to show that

$$\frac{\mu(x+y)}{x+y}-\frac{\mu(x)}{x} \geq 4\log\left(1+\frac{y}{x}\right)\log\log_2(4(x+y)).$$

There are two cases. If $x<1$ then since $x+y\geq 1$,

$$\frac{\mu(x+y)}{x+y}-\frac{\mu(x)}{x}=\frac{\mu(x+y)}{x+y}=4\log(4(x+y))\log\log_2(4(x+y))$$
$$=4\left[\log\left(1+\frac{y}{x}\right)+\log(4x)\right]\log\log_2\ (4(x+y))$$
$$\geq 4\log\left(1+\frac{y}{x}\right)\log\log_2(4(x+y))$$

since $4x\geq 1$, as required. If $x\geq 1$ then

$$\frac{\mu(x+y)}{x+y}-\frac{\mu(x)}{x}=4\log(4(x+y))\log\log_2(4(x+y))-4\log(4x)\log\log_2(4x)$$
$$\geq 4[\log(4(x+y))-\log(4x)]\log\log_2(4(x+y))$$
$$=4\log\left(1+\frac{y}{x}\right)\log\log_2(4(x+y))$$

as required. ∎

Thus, it remains to prove (2.1). We need the following lemma.

(2.3). *Let μ be suitable, let $k>0$ be real, for $0\leq x\leq 1$ let y be a real-valued continuous function of x, and let $I\subseteq[0,1]$ be finite, such that $y(0)\geq 0, y(1)\leq 1$, and for all $h\in[0,1]-I, y$ is differentiable and $\frac{dy}{dx}\geq\frac{1}{k}$ when $x=h$. Then there exists h with $\frac{1}{4}<h<\frac{3}{4}$ such that $h\notin I$ and when $x=h$,*

$$k\frac{dy}{dx}\leq\mu(k)-\mu(ky)-\mu(k(1-y)).$$

Proof. By replacing x,y by $1-x,1-y$ if necessary, we may assume that $y\left(\frac{1}{2}\right)\leq\frac{1}{2}$. Since y is monotone increasing, it follows that $0<y\left(\frac{1}{4}\right)<y\left(\frac{1}{2}\right)\leq\frac{1}{2}$. Also, since $\frac{dy}{dx}\geq\frac{1}{k}$ for $0\leq x\leq 1$ except at finitely many values of x, it follows that $y\left(\frac{1}{4}\right)\geq(4k)^{-1}$. For the same reason and since $y(1)-y(0)\leq 1$, it follows that $k\geq 1$.

For $0<x<1$, define $z=z(x)=-(\log(y))^{-1}$. Since $y\left(\frac{1}{4}\right)\geq(4k)^{-1}$, it follows that

$$z\left(\tfrac{1}{4}\right)\geq-(\log((4k)^{-1}))^{-1}=(\log(4k))^{-1}.$$

Since $y\left(\frac{1}{2}\right)\leq\frac{1}{2}$, it follows that

$$z\left(\tfrac{1}{2}\right)\leq-\left(\log\left(\tfrac{1}{2}\right)\right)^{-1}=(\log 2)^{-1}.$$

Thus, $z\left(\frac{1}{2}\right)/z\left(\frac{1}{4}\right)\leq\log_2(4k)$, and so

$$\log\left(z\left(\tfrac{1}{2}\right)\right)-\log\left(z\left(\tfrac{1}{4}\right)\right)\leq\log(\log_2(4k)).$$

Therefore, there exists h with $\frac{1}{4}<h<\frac{1}{2}$ and with $h\notin I$, such that when $x=h$,

$$\frac{d}{dx}(\log z)\leq 4\log\log_2(4k),$$

that is,

$$\frac{dz}{dx} \leq 4z \log \log_2(4k).$$

But $y = exp(-z^{-1})$, and so $\frac{dy}{dz} = yz^{-2}$. Hence, when $x = h$,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \leq (yz^{-2})(4z \log \log_2(4k)) = 4y(-\log y) \log \log_2(4k).$$

Now when $x = h$, since

$$(4k)^{-1} \leq y\left(\tfrac{1}{4}\right) < y < y\left(\tfrac{1}{2}\right) \leq \tfrac{1}{2},$$

it follows that $k(1-y) \geq ky \geq \frac{1}{4}$; and $ky + k(1-y) = k \geq 1$ as we already saw. From condition (iv) in the definition of "suitable",

$$\mu(ky + k(1-y)) - \mu(ky) - \mu(k(1-y)) \geq$$
$$4ky \log\left(1 + \frac{k(1-y)}{ky}\right) \log \log_2(4(ky + k(1-y))),$$

that is,

$$\mu(k) - \mu(ky) - \mu(k(1-y)) \geq 4ky(-\log y) \log \log_2(4k).$$

It follows that

$$k\frac{dy}{dx} \leq \mu(k) - \mu(ky) - \mu(k(1-y))$$

as required. ∎

If G is a digraph and $A, B \subseteq V(G)$, we denote by $D(A,B)$ or $D_G(A,B)$ the set of edges of G with tail in A and head in B.

(2.4). *Let μ be suitable. Let G be a digraph with $|V(G)| \geq 2$, and let $n \geq 2$ be an integer such that every directed circuit of G has length $\geq n$. Then there is a partition (A,B,C) of $V(G)$ with $A, B \neq V(G)$ and $D(A,B) = \emptyset$ (A or B may be null), such that*

$$|C| \leq \mu\left(\frac{|V(G)|}{n}\right) - \mu\left(\frac{|A|}{n}\right) - \mu\left(\frac{|B|}{n}\right).$$

Proof. If G is not strongly connected, there is a partition A, B of $V(G)$ with $A, B \neq \emptyset$ and with $D(A,B) = \emptyset$; and then $C = \emptyset$ satisfies the assertion of the theorem (by condition (iii) in the definition of "suitable"). We assume, therefore, that G is strongly connected.

(1). *There is a partition $X_1, \ldots, X_n$ of $V(G)$ into non-empty sets such that for $1 \leq i, j \leq n$, if $j > i+1$ then $D(X_i, X_j) = \emptyset$.*

For let $v_0 \in V(G)$. For $1 \leq i \leq n-1$ let X_i be the set of all $v \in V(G)$ such that there is a directed path from v_0 to v and the shortest such path has exactly i edges; and let $X_n = V(G) - (X_1 \cup \ldots \cup X_{n-1})$. Then $X_1, \ldots, X_n$ are mutually disjoint and have union $V(G)$. Moreover, let $1 \le i, j \le n$ with $j > i+1$, and let $u \in X_i$ and $v \in X_j$. Then there is a path P from v_0 to u of length i. If $v \in V(P)$ (this is only possible if $v = v_0$) then u is not adjacent to v since $i+1 \le j-1 < n$ and G has no directed circuit of length $< n$. If $v \notin V(P)$ then $v \ne v_0$, and hence there is no path from v_0 to v of length $< j$; and so again u is not adjacent to v. Thus, $D(X_i, X_j) = \emptyset$. Finally, we must show that $X_1, \ldots, X_n$ are all non-empty. Certainly $X_n \ne \emptyset$ since $v_0 \in X_n$. Let $1 \le i < n$. Since G is strongly connected and $X_1 \ne \emptyset$ from the construction (since $|V(G)| \ge 2$), there exists $u \in X_1 \cup \ldots \cup X_i$ adjacent to some $v \in X_{i+1} \cup \ldots \cup X_n$. But $D(X_h, X_j) = \emptyset$ if $j \ge h+2$, and so $u \in X_i$; and consequently $X_i \ne \emptyset$, as required.

For $x \in [0,1]$, define $y(x)$ as follows. If $x = 0$, let $y(x) = 0$. If $x > 0$, let $i = \lceil nx \rceil$, so that $1 \le i \le n$, and let

$$y(x) = (|X_1| + \ldots + |X_{i-1}| + (nx - i + 1)|X_i|)|V(G)|^{-1}.$$

Then y is a continuous function of x, and $y(0) = 0, y(1) = 1$. Let $I = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$; then y is differentiable in $[0,1] - I$. Let $k = n^{-1}|V(G)|$. For $x \in [0,1] - I$, let $i = \lceil nx \rceil$; then $\frac{dy}{dx} = k^{-1}|X_i|$. By (1), $|X_i| \ge 1$, and so $\frac{dy}{dx} \ge k^{-1}$. By (2.3) there exists $h \in [0,1] - I$ such that when $x = h$,

$$k\frac{dy}{dx} \le \mu(k) - \mu(ky) - \mu(k(1-y)).$$

Let $i = \lceil nh \rceil$; then $1 \le i \le n$, and $k\frac{dy}{dx} = |X_i|$. Hence

$$|X_i| \le \mu(k) - \mu(ky) - \mu(k(1-y))$$

and

$$(|X_1| + \ldots + |X_{i-1}|)|V(G)|^{-1} \le y \le (|X_1| + \ldots + |X_i|)|V(G)|^{-1}.$$

Let $A = X_1 \cup \ldots \cup X_{i-1}$, $B = X_{i+1} \cup \ldots \cup X_n$, $C = X_i$. Then

$$|A| \le y|V(G)| \le |V(G) - B|.$$

Consequently,

$$\mu(ky) = \mu\left(\frac{y|V(G)|}{n}\right) \ge \mu\left(\frac{|A|}{n}\right)$$

since μ is non-decreasing, and

$$\mu(k(1-y)) = \mu\left((1-y)\frac{|V(G)|}{n}\right) \ge \mu\left(\frac{|B|}{n}\right)$$

for the same reason. It follows that

$$|C| = |X_i| \le \mu(k) - \mu(ky) - \mu(k(1-y)) \le \mu\left(\frac{|V(G)|}{n}\right) - \mu\left(\frac{|A|}{n}\right) - \mu\left(\frac{|B|}{n}\right).$$

Finally, $D(A,B) = \emptyset$ from the construction, and $A, B \ne V(G)$ since $C = X_i \ne \emptyset$. The result follows. ∎

If G is a graph or digraph, $G\backslash X$ denotes the graph or digraph obtained from G by deleting X.

Proof of (2.1). We proceed by induction on $|V(G)|$. Let $n = \lceil k^{-1}|V(G)|\rceil$; then every directed circuit of G has length $\geq n$. If G has no directed circuits then taking $X = \emptyset$ satisfies the theorem, since $\mu(k) \geq 0$ by condition (i) in the definition of "suitable". We assume, therefore, that G has a directed circuit, and consequently $|V(G)| \geq n \geq 1$. If $n=1$ then $|V(G)| \leq k$, and taking $X = V(G)$ satisfies the theorem, since $k \leq \mu(k)$ by condition (ii) in the definition of "suitable". We assume, therefore, that $n \geq 2$, and consequently $|V(G)| \geq 2$.

By (2.4), there is a partition (A,B,C) of $V(G)$ with $A,B \neq V(G)$ and $D(A,B) = \emptyset$, such that

$$|C| \leq \mu\left(\frac{|V(G)|}{n}\right) - \mu\left(\frac{|A|}{n}\right) - \mu\left(\frac{|B|}{n}\right).$$

Let $G' = G\backslash(B \cup C)$ and $k' = \frac{|A|}{n}$. If $k' > 0$, then every directed circuit of G' has length $\geq n = k'^{-1}|V(G')|$, and since $|A| < |V(G)|$ it follows from the inductive hypothesis that there exists $Y \subseteq A$ meeting every directed circuit of G', with

$$|Y| \leq \mu(k') = \mu\left(\frac{|A|}{n}\right).$$

The same conclusion holds if $k' = 0$, for then $A = \emptyset$ and we may take $Y = \emptyset$. Similarly, there exists $Z \subseteq B$ with $|Z| \leq \mu\left(\frac{|B|}{n}\right)$ meeting every directed circuit of $G\backslash(A \cup C)$. Since $D(A,B) = \emptyset$, it follows that $X = Y \cup Z \cup C$ meets every directed circuit of G, and

$$|X| \leq |C| + \mu\left(\frac{|A|}{n}\right) + \mu\left(\frac{|B|}{n}\right) \leq \mu\left(\frac{|V(G)|}{n}\right)$$

as required. ∎

3. Remarks

It would be nice to obtain the best possible bound in (1.2), and to pin this down we need a construction of an appropriate digraph. The following construction was found in joint work with Noga Alon.

Let n be a large even integer, and let H be a graph with n vertices such that

(i) every vertex of H has valency 6,

(ii) every circuit of H has length $\geq \frac{4}{5} \log n$,

(iii) for every partition A,B of $V(H)$ into two sets of cardinality $\frac{1}{2}n$, there are at least $\frac{1}{4}n$ edges with one end in A and the other in B.

(Such graphs H exist; for instance by theorem (2.1) on page 120 of [1], applied to the graphs constructed in [2, 3].) Now H is Eulerian; direct the edges of H to obtain a digraph K in which every vertex has outvalency 3. Let G be the "directed line graph" of K; that is, G is a digraph with $V(G) = E(K)$, and for $e, f \in V(G) = E(K)$, e is adjacent to f in G if the head of e in K equals the tail of f in K.

We see that $|V(G)| = 3|V(H)| = 3n$, and every directed circuit of G has length $\geq \frac{4}{5}\log n$. Consequently, every fractional circuit packing in G has value $\leq k$ where $k = 15n/(4\log n)$.

Let $X \subseteq V(G)$, meeting every directed circuit of G. Then $X \subseteq E(K)$, and X meets every directed circuit of K. Consequently, $K\backslash X$ has no directed circuits, and so we may order the vertices of K as $v_1,\ldots,v_n$ so that for $1 \leq i < j \leq n$, v_i is not adjacent to v_j in $K\backslash X$. Let $A = \{v_1,\ldots,v_{\frac{1}{2}n}\}$, $B = \{v_{\frac{1}{2}n+1},\ldots,v_n\}$. Then $D_K(A,B) \subseteq X$, and since

$$|D_K(B,A)| = |D_K(A,B)|$$

because K is an Eulerian digraph, it follows that there are at most $2|X|$ edges of H with one end in A and the other in B. By property (iii) above, $2|X| \geq \frac{1}{4}n$, and so

$$|X| \geq \frac{n}{8} \geq \frac{1}{30} k \log k$$

for n sufficiently large. This shows that the bound in (1.2) is best possible except for the $\log\log_2(4k)$ term and the multiplicative constant.

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